

$K_{l3\gamma}^+$ decays revisited: branching ratios and T -odd momenta correlations

I.B. Khriplovich¹ and A.S. Rudenko²
Budker Institute of Nuclear Physics,
630090 Novosibirsk, Russia

Abstract

We calculate the branching ratios of the $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ ($l = e, \mu$) decays, and the T -odd triple momenta correlations $\xi = \vec{q} \cdot [\vec{p}_l \times \vec{p}_\pi] / M_K^3$, due to the electromagnetic final state interaction, in these processes. The contributions on the order of ω^{-1} and ω^0 to the corresponding amplitudes are treated exactly. For the branching ratios, the corrections on the order of ω are estimated and demonstrated to be small. We compare the results with those of other authors. In some cases our results differ considerably from the previous ones.

1. The $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ ($l = e, \mu$) decays were earlier studied theoretically in Refs. [1, 2, 3]. Therein the branching ratios of these decays were calculated. Besides, in Ref. [2] the T -odd triple momenta correlations $\xi = \vec{q} \cdot [\vec{p}_l \times \vec{p}_\pi] / M_K^3$ were considered, as induced by the electromagnetic final state interactions; here and below M_K is the kaon mass, \vec{q} , \vec{p}_l , \vec{p}_π are the momenta of γ , l^+ , π^0 , respectively. In principle, these triple correlations can be used to probe new CP -odd effects beyond the Standard Model, which could also contribute to them.

Here we calculate anew these effects. Our results confirm essentially some previous results and disagree considerably with other ones.

In the theoretical analysis of radiative effects in the discussed processes, the treatment of the accompanying radiation, which gives the effects on the order of ω^{-1} and ω^0 (the last ones originate from the radiation due to the lepton magnetic moment), is straightforward (here and below ω is the photon energy). As to the structure radiation contribution on the order of ω^0 , it is also under control, due in fact to the gauge invariance [4]. The contributions on the order of ω (and higher) depend directly on the photon field strength $F_{\mu\nu}$ (and its derivatives), and cannot be fixed in a model-independent way. We assume that the corrections on the order of ω and higher are relatively small. And indeed, more quantitative arguments presented below demonstrate that such contributions into the discussed branching ratios do not exceed few percent.

2. At the tree level, the $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ decay is described by the Feynman graphs in Fig. 1.

The matrix elements for diagrams 1a and 1b look as follows:

$$\begin{aligned} M_{1a} &= \frac{G}{\sqrt{2}} \sin \theta_c e \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \frac{\hat{p}_l + \hat{q} - m_l}{2p_{lq}} \hat{e}^* v_l [f_+(t) \cdot (p_K + p_\pi)_\alpha + f_-(t) \cdot (p_K - p_\pi)_\alpha] \\ &= \frac{G}{\sqrt{2}} \sin \theta_c e \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \left(\frac{2p_l e^* + \hat{q} \hat{e}^*}{2p_{lq}} \right) v_l [f_+(t) \cdot (p_K + p_\pi)_\alpha + f_-(t) \cdot (p_K - p_\pi)_\alpha], \quad (1) \end{aligned}$$

¹khriplovich@inp.nsk.su

²a.s.rudenko@inp.nsk.su

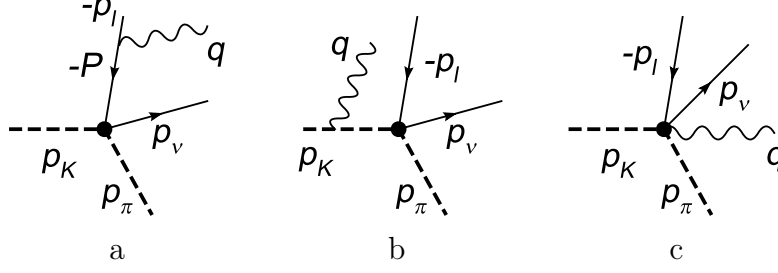


Figure 1: The tree diagrams

$$M_{1b} = -\frac{G}{\sqrt{2}} \sin \theta_c e \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l [f_+(t') \cdot (p_K - q + p_\pi)_\alpha + f_-(t') \cdot (p_K - q - p_\pi)_\alpha] \frac{p_K e^*}{p_K q}; \quad (2)$$

here G is the Fermi coupling constant, θ_c is the Cabibbo angle, e is the elementary charge ($e > 0$), $t = (p_K - p_\pi)^2$, $t' = (p_K - q - p_\pi)^2$; the lower indices attached to the matrix elements match the corresponding Feynman diagrams.

Usually the dependence of the form factors f_+ and f_- on the momentum transfer t is described by formula

$$f_\pm(t) = f_\pm(0) \left(1 + \lambda_\pm \frac{t}{m_\pi^2} \right). \quad (3)$$

The experimental data are adequately described by Eq. (3) with $\lambda_+ \approx 0.03$ for $l = \mu$ and $l = e$; $\lambda_- = 0$ for $l = \mu$ [5]; λ_- for $l = e$ is unknown, but one may assume that it is also close to zero.

In the $K_{l3\gamma}^+$ decays the ratio $\lambda_\pm t/m_\pi^2$ is small, $\lambda_\pm t/m_\pi^2 \lesssim 0.1$, so one can put $f_\pm(t) = f_\pm(0)$. Since the ratio $\xi(0) = f_-(0)/f_+(0) \sim 0.1$ is also small [6], one can neglect $f_-(0)$ with the same accuracy.

Thus, our expressions (1), (2) simplify to

$$M_{1a} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \cdot (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \left(\frac{p_l e^*}{p_l q} + \frac{\hat{q} \hat{e}^*}{2 p_l q} \right) v_l, \quad (4)$$

$$M_{1b} = -\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \cdot (p_K - q + p_\pi)_\alpha \frac{p_K e^*}{p_K q} \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l. \quad (5)$$

However, the sum of diagrams 1a and 1b is not gauge invariant: it does not vanish under the substitution $e^* \rightarrow q$. To restore the gauge invariance, one should add the third diagram where a photon is directly emitted from the vertex (see Fig. 1c). This contact amplitude has no single-particle intermediate states, and therefore is on the order of ω^0 and higher. The contribution $\sim \omega^0$, as derived with the Low technique [4], is

$$M_{1c} = -\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \cdot e_\alpha^* \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l. \quad (6)$$

Thus, the model-independent gauge invariant tree amplitude of $K_{l3\gamma}^+$ decay, including only terms on the order of ω^{-1} and ω^0 (but all of them!), is

$$M_1 = M_{1a} + M_{1b} + M_{1c} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \left\{ (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l \left(\frac{p_l e^*}{p_l q} - \frac{p_K e^*}{p_K q} \right) + (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \frac{\hat{q} \hat{e}^*}{2 p_l q} v_l + \left(\frac{p_K e^*}{p_K q} q_\alpha - e_\alpha^* \right) \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l \right\}. \quad (7)$$

This expression agrees with the corresponding formulas in Ref. [1] and formula (6) in Ref. [2] (if our $f_+(0)$ is set to its $SU(3)$ value $f_+(0) = 1/\sqrt{2}$).

It is convenient to present the amplitude (7) as a sum of gauge invariant contributions. They are the "infrared" term M_{IR} corresponding to the sum of the amplitudes of accompanying radiation by the kaon and lepton (independent of the lepton magnetic moment), the magnetic term M_{mag} which is the amplitude of spin-dependent accompanying radiation of the lepton magnetic moment, and the Low term M_{Low} :

$$M_{IR} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l \left(\frac{p_l e^*}{p_l q} - \frac{p_K e^*}{p_K q} \right), \quad (8)$$

$$M_{mag} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \frac{\hat{q} \hat{e}^*}{2 p_l q} v_l, \quad (9)$$

$$M_{Low} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \left(\frac{p_K e^*}{p_K q} q_\alpha - e_\alpha^* \right) \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_l. \quad (10)$$

The results of calculation for $K_{l3\gamma}^+$ branching ratios are presented in Table 1; here the following cuts in the kaon rest frame are used: $\omega \geq 30$ MeV and $\theta_{l\gamma} \geq 20^\circ$.

| | $l = \mu$ | $l = e$ |
|---------------------|---|---|
| Bijnens et al. [1] | 1.9×10^{-5} | 2.8×10^{-4} |
| Braguta et al. [2] | 2.15×10^{-5} | 3.18×10^{-4} |
| present work | 1.81×10^{-5} | 2.72×10^{-4} |
| experimental values | $(2.4 \pm 0.5 \pm 0.6) \times 10^{-5}$ [7] $(1.58 \pm 0.46 \pm 0.08) \times 10^{-5}$ [8] | $(3.06 \pm 0.09 \pm 0.14) \times 10^{-4}$ [9] |

Table 1: Branching ratio of $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ decays

The accuracy of our results can be estimated as follows. The leading corrections to them are due to the structure radiation from the hadronic vertex. They are proportional to the photon field strength, i.e. are on the order of ω . There are good reasons to believe that these corrections are less than the Low structure amplitudes which are on the order of ω^0 . The Low contributions (including of course their interference with the accompanying radiation) to the discussed $K^+ \rightarrow \pi^0 \mu^+ \nu_\mu \gamma$ and $K^+ \rightarrow \pi^0 e^+ \nu_e \gamma$ branching ratios, according to our calculations, constitute -0.24×10^{-5} and -0.12×10^{-4} , respectively. Thus, we estimate the accuracy of our results as $\pm 0.2 \times 10^{-5}$ and $\pm 0.1 \times 10^{-4}$, correspondingly. Let us note also that corrections to the quoted results derived in Ref. [1, 3] in the chiral perturbation theory are of similar magnitude.

As mentioned, additional corrections on the level of 10 – 15% to the branching ratios originate from our neglect of the form factor $f_-(t)$ and of the t -dependence of $f_+(t)$ (the last correction is certainly positive). As to the relative accuracy of our numerical integration over phase space of final particles, it is about 1%.

To compare properly our results with those of Refs. [1, 2] one should keep in mind that now the experimental values of some quantities are known with better accuracy. Indeed, we use $\sin \theta_c f_+(0) = 0.217/\sqrt{2}$ in our calculation, and, as far as we can see, in Refs. [1, 2]

the corresponding value is $0.22/\sqrt{2}$. Substitution of one of these values for another alters the results by about 3%.

Thus, our results for the branching ratios agree reasonably well with those of Ref. [1]. There is however some disagreement between our results and those of Ref. [2].

3. The T -odd triple momenta correlation $\xi = \vec{q} \cdot [\vec{p}_l \times \vec{p}_\pi]/M_K^3$ in the $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ decays arise from the interference term $2\text{Re}(M_1^* A_2)$ in the decay rate; here M_1 is the tree amplitude and A_2 is the anti-Hermitian part of the one-loop diagrams presented in Fig. 2.

One can easily demonstrate that A_2 is generated only by attaching the intermediate photon to diagram 1a. The on-mass-shell intermediate particles in Fig. 2 are marked by crosses.

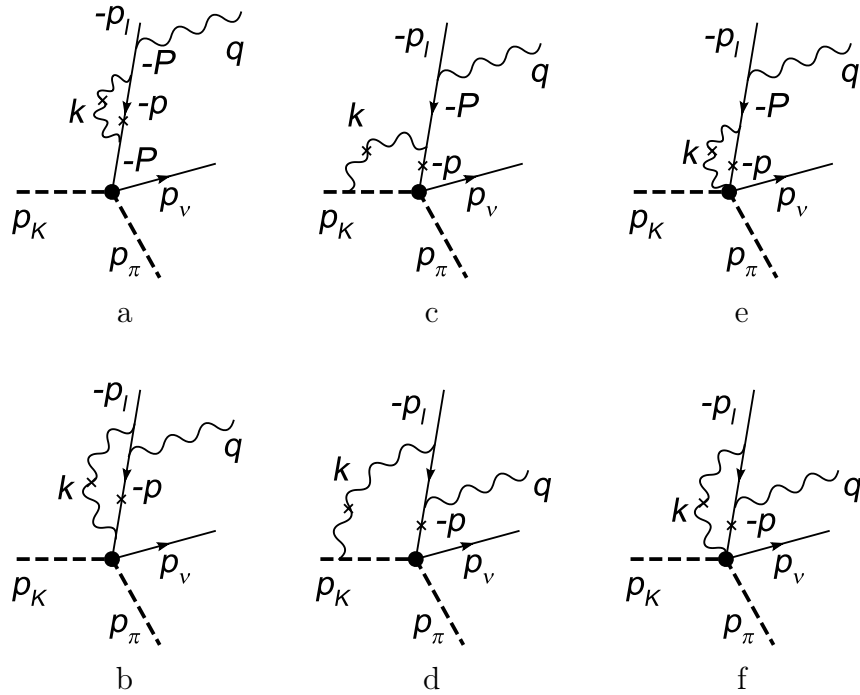


Figure 2: The one-loop diagrams

The anti-Hermitian part of the sum of one-loop diagrams is written as

$$A_2 = \frac{i}{8\pi^2} \sum_n M_{fn} M_{in}^*. \quad (11)$$

The Compton amplitude entering this expression is (see Fig. 3)

$$M_{fn} = M_{3a} + M_{3b} = e^2 \bar{v}_p \hat{e}_k \frac{\hat{P} - m_l}{2p_l q} \hat{e}^* v_l + e^2 \bar{v}_p \hat{e}^* \frac{\hat{p}_l - \hat{k} - m_l}{-2p_l k} \hat{e}_k v_l. \quad (12)$$

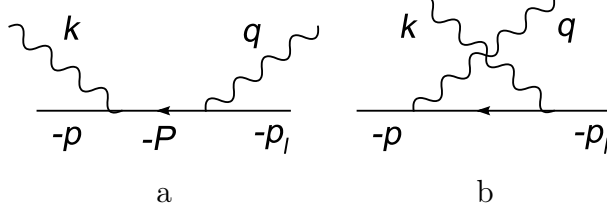


Figure 3: The Compton scattering diagrams

As to M_{in}^* , it is the same tree amplitude (7), up to the change of some notations:

$$M_{in}^* = M_{ni} = M_1 \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \left\{ (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_p \left(\frac{p e_k^*}{p_l q} - \frac{p_K e_k^*}{p_K k} \right) \right. \\ \left. + (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \frac{\hat{k} \hat{e}_k^*}{2 p_l q} v_p + \left(\frac{p_K e_k^*}{p_K k} k_\alpha - e_{k\alpha}^* \right) \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) v_p \right\}. \quad (13)$$

The sum over n in formula (11) includes the summation over the polarizations of intermediate particles, and the integral over the phase space with

$$d\rho = \frac{d^3 k}{2\omega_k} \frac{d^3 p}{2E_p} \delta^{(4)}(p + k - P). \quad (14)$$

The details of the calculation of the T -odd correlation $\xi = \vec{q} \cdot [\vec{p}_l \times \vec{p}_\pi] / M_K^3$ are given in Appendix. Here we wish to note only that our result for ξ -odd term $|M|_{odd}^2$ in the interference $2\text{Re}(M_1^* A_2)$ agrees with formula (12) in Ref. [2] up to the factor $1/\sqrt{2}$ which is obviously omitted therein (it follows directly from formulas (6) and (10) of Ref. [2]).

In fact, what is really measured experimentally, is not the T -odd triple momentum correlation ξ by itself, but the asymmetry

$$A_\xi = \frac{N_+ - N_-}{N_+ + N_-} = \frac{\int (|M_1|^2 + |M|_{odd}^2) d\Phi_{\xi>0} - \int (|M_1|^2 + |M|_{odd}^2) d\Phi_{\xi<0}}{\int (|M_1|^2 + |M|_{odd}^2) d\Phi_{\xi>0} + \int (|M_1|^2 + |M|_{odd}^2) d\Phi_{\xi<0}} \\ = \frac{\int |M|_{odd}^2 d\Phi_{\xi>0}}{\int |M_1|^2 d\Phi_{\xi>0}} \quad (15)$$

induced by this correlation; here N_+ and N_- are the numbers of events with $\xi > 0$ and $\xi < 0$, and integration is performed over the phase space of the final particles.

The results for the asymmetry A_ξ are presented in Table 2. The relative accuracy of our numerical integration is about 1%, as it was the case with the branching ratios.

Here however, as distinct from the problem of branching ratios, the contribution of the Low term is large, quite comparable numerically to the contributions of the accompanying radiation which are on the order of ω^{-1} and ω^0 . So, here it is difficult to estimate reliably the relative magnitude of the structure radiation contribution proportional to ω , i.e. to estimate reliably the true accuracy of thus derived results for the asymmetry A_ξ . We note here that corrections to the value of the discussed correlation, derived in Ref. [10] within the chiral perturbation theory, are very small.

Our results for A_ξ exceed those of Ref. [2], obtained in the same approximation, by a factor of 1.5 – 2. If one included the factor $1/\sqrt{2}$, probably lost in the calculations of Ref. [2] (see the remark above), it would make the disagreement even worse.

| | $l = \mu$ | $l = e$ |
|---------------------|---|--|
| Braguta et al. [2] | 1.14×10^{-4} $\omega > 30 \text{ MeV}, \theta_{l\gamma} > 20^\circ$ | -0.59×10^{-4} $\omega > 30 \text{ MeV}, \theta_{l\gamma} > 20^\circ$ |
| present work | 2.38×10^{-4} $\omega \geq 30 \text{ MeV}, \theta_{l\gamma} \geq 20^\circ$ | -0.93×10^{-4} $\omega \geq 30 \text{ MeV}, \theta_{l\gamma} \geq 20^\circ$ |
| experimental values | -0.03 ± 0.13 [11] $5 < \omega < 30 \text{ MeV}$ | -0.015 ± 0.021 [9] $\omega > 10 \text{ MeV}, 0.6 < \cos \theta_{l\gamma} < 0.9$ |
| present work | 0.50×10^{-4} $5 < \omega < 30 \text{ MeV}$ | -0.30×10^{-4} $\omega > 10 \text{ MeV}, 0.6 < \cos \theta_{l\gamma} < 0.9$ |

Table 2: A_ξ in $K^+ \rightarrow \pi^0 l^+ \nu_l \gamma$ decay

The asymmetry A_ξ was measured experimentally [9, 11], but cuts imposed therein differ from those used in Ref. [2]. Therefore, we have calculated also the asymmetry for the corresponding kinematical regions (see the last line in Table 2).

As distinct from the situation with the branching ratios, all the theoretical results for the triple correlations are unfortunately far away from the real experimental sensitivity.

Acknowledgements

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Appendix A

In this section we give the list of the integrals that contribute to the A_2 :

$$\int d\rho = a_0 = \frac{\pi}{2} \left(1 - \frac{m_l^2}{P^2} \right); \quad (\text{A.1})$$

$$\int k_\mu d\rho = a_P P_\mu, \text{ where } a_P = \frac{\pi}{4} \left(1 - \frac{m_l^2}{P^2} \right)^2; \quad (\text{A.2})$$

$$\int \frac{1}{p_l k} d\rho = b_0 = \frac{\pi}{2p_l q} \ln \left(\frac{P^2}{m_l^2} \right); \quad (\text{A.3})$$

$$\int \frac{k_\mu}{p_l k} d\rho = B_{1\mu} = b_l p_{l\mu} + b_P P_\mu, \quad (\text{A.4})$$

here b_l and b_P are the solutions of the system of equations

$$\begin{aligned} b_l m_l^2 + b_P(p_l P) &= a_0, \\ b_l(p_l P) + b_P P^2 &= b_0(p_l q); \end{aligned}$$

$$\int \frac{k_\mu k_\nu}{p_l k} d\rho = B_{2\mu\nu} = b_2 g_{\mu\nu} + b_{ll} p_{l\mu} p_{l\nu} + b_{PP} P_\mu P_\nu + b_{lP}(p_{l\mu} P_\nu + P_\mu p_{l\nu}), \quad (\text{A.5})$$

here b_2 , b_{ll} , b_{PP} , and b_{lP} are the solutions of the system of equations

$$\begin{aligned} 4b_2 + b_{ll} m_l^2 + b_{PP} P^2 + 2b_{lP}(p_l P) &= 0, \\ b_2 + b_{ll} m_l^2 + b_{lP}(p_l P) &= 0, \\ b_{PP}(p_l P) + b_{lP} m_l^2 &= a_P, \\ b_2 + b_{PP} P^2 + b_{lP}(p_l P) &= b_P(p_l q); \end{aligned}$$

$$\int \frac{1}{p_K k} d\rho = c_0 = \frac{\pi}{2} \frac{1}{\sqrt{(p_K P)^2 - m_K^2 P^2}} \ln \left(\frac{p_K P + \sqrt{(p_K P)^2 - m_K^2 P^2}}{p_K P - \sqrt{(p_K P)^2 - m_K^2 P^2}} \right); \quad (\text{A.6})$$

$$\int \frac{k_\mu}{p_K k} d\rho = C_{1\mu} = c_K p_{K\mu} + c_P P_\mu, \quad (\text{A.7})$$

here c_K and c_P are the solutions of the system of equations

$$\begin{aligned} c_K m_K^2 + c_P(p_K P) &= a_0, \\ c_K(p_K P) + c_P P^2 &= c_0(p_l q); \end{aligned}$$

$$\int \frac{1}{(p_l k)(p_K k)} d\rho = d_0 = \frac{\pi}{2p_l q} \frac{1}{\sqrt{(p_K p_l)^2 - m_K^2 m_l^2}} \ln \left(\frac{p_K p_l + \sqrt{(p_K p_l)^2 - m_K^2 m_l^2}}{p_K p_l - \sqrt{(p_K p_l)^2 - m_K^2 m_l^2}} \right); \quad (\text{A.8})$$

$$\int \frac{k_\mu}{(p_l k)(p_K k)} d\rho = D_{1\mu} = d_K p_{K\mu} + d_l p_{l\mu} + d_P P_\mu, \quad (\text{A.9})$$

here d_K , d_l , and d_P are the solutions of the system of equations

$$\begin{aligned} d_K m_K^2 + d_l(p_K p_l) + d_P(p_K P) &= b_0, \\ d_K(p_K p_l) + d_l m_l^2 + d_P(p_l P) &= c_0, \\ d_K(p_K P) + d_l(p_l P) + d_P P^2 &= d_0(p_l q); \end{aligned}$$

$$\begin{aligned} \int \frac{k_\mu k_\nu}{(p_l k)(p_K k)} d\rho = D_{2\mu\nu} &= d_2 g_{\mu\nu} + d_{KK} p_{K\mu} p_{K\nu} + d_{ll} p_{l\mu} p_{l\nu} + d_{PP} P_\mu P_\nu \\ &+ d_{Kl}(p_{K\mu} p_{l\nu} + p_{l\mu} p_{K\nu}) + d_{KP}(p_{K\mu} P_\nu + P_\mu p_{K\nu}) + d_{lP}(p_{l\mu} P_\nu + P_\mu p_{l\nu}), \end{aligned} \quad (\text{A.10})$$

here d_2 , d_{KK} , d_{ll} , d_{PP} , d_{Kl} , d_{KP} , and d_{lP} are the solutions of the system of equations

$$\begin{aligned}
4d_2 + d_{KK}m_K^2 + d_{ll}m_l^2 + d_{PP}P^2 + 2d_{Kl}(p_K p_l) + 2d_{KP}(p_K P) + 2d_{lP}(p_l P) &= 0, \\
d_2 + d_{ll}m_l^2 + d_{Kl}(p_K p_l) + d_{lP}(p_l P) &= 0, \\
d_{KK}(p_K p_l) + d_{Kl}m_l^2 + d_{KP}(p_l P) &= c_K, \\
d_{PP}(p_l P) + d_{KP}(p_K p_l) + d_{lP}m_l^2 &= c_P, \\
d_2 + d_{KK}m_K^2 + d_{Kl}(p_K p_l) + d_{KP}(p_K P) &= 0, \\
d_{PP}(p_K P) + d_{KP}m_K^2 + d_{lP}(p_K p_l) &= b_P, \\
d_2 + d_{PP}P^2 + d_{KP}(p_K P) + d_{lP}(p_l P) &= d_P(p_l q).
\end{aligned}$$

All these expressions agree with the analogous ones in Ref. [2].

Appendix B

The A_2 can be expressed as follows:

$$\begin{aligned}
A_2 &= \frac{i}{8\pi^2} \sum_n M_{fn} M_{in}^* = \frac{i}{8\pi^2} \sum_n (M_{3a} + M_{3b})(M_{IR} + M_{mag} + M_{Low}) \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} \\
&= A_{3a-IR} + A_{3a-mag} + A_{3a-Low} + A_{3b-IR} + A_{3b-mag} + A_{3b-Low}, \quad (\text{B.1})
\end{aligned}$$

where

$$\begin{aligned}
A_{3a-IR} &= \frac{i}{8\pi^2} \sum_n M_{3a} M_{IR} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \left[c_0(\hat{P} - m_l) \hat{p}_K - \hat{C}_1 \hat{p}_K + \frac{m_l}{p_l q} (a_0(\hat{P} - m_l) - a_P \hat{P}) \right] \frac{\hat{P} - m_l}{2p_l q} \hat{e}^* v_l; \quad (\text{B.2})
\end{aligned}$$

$$\begin{aligned}
A_{3a-mag} &= \frac{i}{8\pi^2} \sum_n M_{3a} M_{mag} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \left[2a_P(P^2 + 2m_l \hat{P}) \right] \frac{\hat{P} - m_l}{(2p_l q)^2} \hat{e}^* v_l; \quad (\text{B.3})
\end{aligned}$$

$$\begin{aligned}
A_{3a-Low} &= \frac{i}{8\pi^2} \sum_n M_{3a} M_{Low} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \left[a_0(\hat{P} - m_l) \gamma_\alpha - a_P \hat{P} \gamma_\alpha - C_{1\alpha}(\hat{P} - m_l) \hat{p}_K \right] \frac{\hat{P} - m_l}{2p_l q} \hat{e}^* v_l; \quad (\text{B.4})
\end{aligned}$$

$$\begin{aligned}
A_{3b-IR} &= \frac{i}{8\pi^2} \sum_n M_{3b} M_{IR} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \frac{1}{2} \left[(\hat{P} - m_l) \hat{e}^* \left(\frac{1}{p_l q} ((\hat{p}_l - m_l)(b_0 \hat{P} - \hat{B}_1) - \hat{B}_1 \hat{P}) - d_0(\hat{p}_l - m_l) \hat{p}_K + \hat{D}_1 \hat{p}_K \right) \right. \\
&\quad \left. - \frac{1}{p_l q} (\hat{B}_1 \hat{e}^* (\hat{p}_l - m_l) \hat{P} - B_{2\mu\nu} \gamma_\mu \hat{e}^* (\hat{p}_l - m_l) \gamma_\nu - B_{2\mu\nu} \gamma_\mu \hat{e}^* \gamma_\nu \hat{P}) \right. \\
&\quad \left. + \hat{D}_1 \hat{e}^* (\hat{p}_l - m_l) \hat{p}_K - D_{2\mu\nu} \gamma_\mu \hat{e}^* \gamma_\nu \hat{p}_K \right] v_l; \quad (B.5)
\end{aligned}$$

$$\begin{aligned}
A_{3b-mag} &= \frac{i}{8\pi^2} \sum_n M_{3b} M_{mag} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) (p_K + p_\pi)_\alpha \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \frac{1}{2p_l q} \left[B_{2\mu\nu} \gamma_\mu \hat{p}_l \hat{e}^* \gamma_\nu + 4m_l B_{2\mu\nu} \gamma_\mu e_\nu^* - \hat{B}_1 \hat{p}_l \hat{e}^* \hat{P} - 4m_l \hat{B}_1 (p_l e^*) - m_l^2 \hat{B}_1 \hat{e}^* \right] v_l; \quad (B.6)
\end{aligned}$$

$$\begin{aligned}
A_{3b-Low} &= \frac{i}{8\pi^2} \sum_n M_{3b} M_{Low} \Big|_{\substack{q \rightarrow k, \\ p_l \rightarrow p, \\ e \rightarrow e_k}} = \frac{ie^2}{8\pi^2} \frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \bar{u}_\nu \gamma_\alpha (1 + \gamma_5) \\
&\times \frac{1}{2} \left[(\hat{P} - m_l) \hat{e}^* ((\hat{p}_l - m_l)(D_{1\alpha} \hat{p}_K - b_0 \gamma_\alpha) - D_{2\alpha\mu} \gamma_\mu \hat{p}_K + \hat{B}_1 \gamma_\alpha) \right. \\
&\quad \left. + \hat{B}_1 \hat{e}^* (\hat{p}_l - m_l) \gamma_\alpha - B_{2\mu\nu} \gamma_\mu \hat{e}^* \gamma_\nu \gamma_\alpha \right] v_l. \quad (B.7)
\end{aligned}$$

Appendix C

The interference term $2\text{Re}(M_1^* A_2)$ can be expressed as follows:

$$\begin{aligned}
2\text{Re}(M_1^* A_2) &= 2\text{Re}((M_{IR}^* + M_{mag}^* + M_{Low}^*) \\
&\times (A_{3a-IR} + A_{3a-mag} + A_{3a-Low} + A_{3b-IR} + A_{3b-mag} + A_{3b-Low})). \quad (C.1)
\end{aligned}$$

We keep here only terms odd in ξ . Let the $|M|_{IR-3a-IR}^2$ be the ξ -odd terms in $2\text{Re}(M_{IR}^* A_{3a-IR})$, the $|M|_{IR-3a-mag}^2$ be the ξ -odd terms in $2\text{Re}(M_{IR}^* A_{3a-mag})$, etc.

Therefore, the ξ -odd term $|M|_{odd}^2$ in the interference term $2\text{Re}(M_1^* A_2)$ is

$$\begin{aligned}
|M|_{odd}^2 &= |M|_{IR-3a-IR}^2 + |M|_{IR-3a-mag}^2 + |M|_{IR-3a-Low}^2 + |M|_{IR-3b-IR}^2 \\
&\quad + |M|_{IR-3b-mag}^2 + |M|_{IR-3b-Low}^2 + |M|_{mag-3a-IR}^2 + |M|_{mag-3a-mag}^2 \\
&\quad + |M|_{mag-3a-Low}^2 + |M|_{mag-3b-IR}^2 + |M|_{mag-3b-mag}^2 + |M|_{mag-3b-Low}^2 \\
&\quad + |M|_{Low-3a-IR}^2 + |M|_{Low-3a-mag}^2 + |M|_{Low-3a-Low}^2 + |M|_{Low-3b-IR}^2 \\
&\quad + |M|_{Low-3b-mag}^2 + |M|_{Low-3b-Low}^2, \quad (C.2)
\end{aligned}$$

where

$$\begin{aligned}
|M|_{IR-3a-IR}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q) (p_l q)^2} \\
&\times (3m_K^2 - m_\pi^2 - 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q)) \\
&\times ((2a_0 - a_P)m_l^2 + c_K m_K^2 (p_l q) - c_P m_l^2 (p_K q) + 2(c_P - c_0)(p_K p_l + p_K q)(p_l q));
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
|M|_{IR-3a-mag}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q) (p_l q)^2} \\
&\times a_P (m_l^2 - 2p_l q) (3m_K^2 - m_\pi^2 - 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q));
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
|M|_{IR-3a-Low}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{2e^2 \xi m_K^4}{\pi^2 (p_K q) (p_l q)^2} \\
&\times ((c_K + 2c_P)((p_l q)^2 m_K^2 + (p_K q)^2 m_l^2 + (p_K p_l)(p_K q)m_l^2) \\
&+ 2c_K(p_K p_l + p_K q)(p_l q)m_K^2 - c_P(p_K p_l + p_K q)(p_l q)(2p_l q + m_l^2) \\
&- 2c_K((p_K p_l)^2 + (p_K q)^2)(p_l q) - 4c_K(p_K p_l)(p_K q)(p_l q) \\
&+ (a_0 + a_P)(p_l q - 2p_K q)m_l^2 + 2(a_P - a_0)(p_l q)^2);
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
|M|_{IR-3b-IR}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q) (p_l q)} \\
&\times (-3m_K^2 + m_\pi^2 + 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q)) \\
&\times (2b_2 + (2b_0 - 5b_l - 5b_P + b_{ll} + 4b_{lP} + 3b_{PP})m_l^2 \\
&+ 2(b_0 - b_l - 3b_P + b_{lP} + 2b_{PP})(p_l q) + (d_K - 2d_{KP})(p_l q)m_K^2 \\
&+ (-d_l - 3d_P + 2d_{lP} + 2d_{PP})m_l^2(p_K q) + 2d_{KP}(p_K p_l)(p_K q) \\
&+ 2(-d_0 + d_l + 2d_P - d_{lP} - d_{PP})(p_K p_l)(p_l q));
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
|M|_{IR-3b-mag}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q) (p_l q)} \\
&\times (-3m_K^2 + m_\pi^2 + 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q)) \\
&\times ((b_l + b_P - 2b_{lP} - 2b_{PP})m_l^2 + 2(b_P - b_{PP})(p_l q));
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
|M|_{IR-3b-Low}^2 = & \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{-e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)} \\
& \times (-2d_{KK}(p_l q)m_K^4 + 2(d_{Kl} - 2d_{PP})(p_l q)^2 m_K^2 \\
& + (-d_{KK} + 4d_{Kl} + 4d_{KP})(p_K q)m_l^2 m_K^2 + 2(d_{ll} - d_{PP})(p_l q)m_l^2 m_K^2 \\
& + 2(2d_K + d_{KK} - 4d_{Kl} - 2d_{KP})(p_K p_l)(p_l q)m_K^2 + 2d_{KK}(p_K q)(p_l q)m_K^2 \\
& + 2(-d_{Kl} - 3d_{KP})(p_K q)^2 m_l^2 + 4(-2b_P + b_{lP} + b_{PP})(p_l q)^2 \\
& + 4(-d_l - d_P + d_{ll} + 2d_{lP} + d_{PP})(p_K p_l)(p_l q)^2 \\
& + (-d_{ll} - 4d_{lP} - 3d_{PP})(p_K q)m_l^4 + 2(-6b_P + 4b_{lP} + 4b_{PP} - 3d_2)(p_K q)m_l^2 \\
& + 2(d_K - 2d_l + 2d_P - d_{Kl} - 2d_{KP} + 2d_{ll} + 4d_{lP} + 2d_{PP})(p_K p_l)(p_K q)m_l^2 \\
& + 2(-3b_P + b_{ll} + 2b_{lP} + b_{PP})(p_l q)m_l^2 \\
& + 4(-d_K + 2d_l + d_{Kl} + d_{KP} - 2d_{ll} - 2d_{lP})(p_K p_l)^2 (p_l q) \\
& + 4b_2(p_l q) + 2(-d_l - d_P + d_{ll} + 2d_{lP} + d_{PP})(p_K p_l)(p_l q)m_l^2 \\
& + 4(4b_P - 2b_{lP} - 2b_{PP} + d_2)(p_K p_l)(p_l q) \\
& + 2(-d_{ll} - 3d_{lP} - 3d_{PP})(p_K q)(p_l q)m_l^2 \\
& + 4(-d_K + d_{Kl} + 2d_{KP} + 2d_{PP})(p_K p_l)(p_K q)(p_l q) \\
& + 4(b_0 - b_l)(p_K q)m_l^2 + 2(b_0 - 3b_l)(p_l q)m_l^2 + 4(b_0 - b_l)(p_l q - 2p_K p_l)(p_l q));
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
|M|_{mag-3a-IR}^2 = & \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_l q)^2} (c_P(m_l^2 - 2p_l q) + 2c_0(p_l q)) \\
& \times (3m_K^2 - m_\pi^2 - 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q));
\end{aligned} \tag{C.9}$$

$$|M|_{mag-3a-mag}^2 = 0; \tag{C.10}$$

$$\begin{aligned}
|M|_{mag-3a-Low}^2 = & \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{-2e^2 \xi m_K^4}{\pi^2 (p_l q)^2} \\
& \times (2c_K(p_l q)m_K^2 - 2c_P(p_l q)^2 + (c_K + 2c_P)(p_K p_l + p_K q)m_l^2 \\
& - c_P(p_l q)m_l^2 - 2c_K(p_K p_l + p_K q)(p_l q) + 4(a_P - a_0)(p_l q) - 2(a_0 + a_P)m_l^2);
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
|M|_{mag-3b-IR}^2 = & \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_l q)} \\
& \times (3m_K^2 - m_\pi^2 - 2(p_K p_l - p_K p_\pi + p_K q + p_l p_\pi + p_\pi q)) \\
& \times (d_{KK}m_K^2 + (d_{ll} + 2d_{lP} + d_{PP} - d_l - d_P)m_l^2 + 2(d_{Kl} + d_{KP} - d_K)(p_K p_l));
\end{aligned} \tag{C.12}$$

$$|M|_{mag-3b-mag}^2 = 0; \tag{C.13}$$

$$\begin{aligned}
|M|_{mag-3b-Low}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{-e^2 \xi m_K^4}{\pi^2 (p_l q)} \\
&\times ((d_{ll} + 2d_{lP} + d_{PP})m_l^4 + 4(2b_l + 2b_P - 2b_{ll} - 4b_{lP} - 2b_{PP} + d_2)m_l^2 \\
&+ d_{KK}m_K^2 m_l^2 + 2(-d_K - 2d_l - 2d_P + d_{Kl} + d_{KP})(p_K p_l)m_l^2 \\
&+ 2(d_{Kl} + 2d_{KP} + 2d_{ll} + 4d_{lP} + 2d_{PP})(p_K q)m_l^2 \\
&+ 2(d_{lP} + d_{PP})(p_l q)m_l^2 - 8b_2 + 4b_0(m_l^2 - 2p_l q) \\
&- 2(d_{KK} + 2d_{Kl} + 2d_{KP})(p_l q)m_K^2 + 8(b_l + 2b_P - b_{lP} - b_{PP})(p_l q) \\
&+ 4(d_K + 2d_l + 2d_P - d_{Kl} - d_{KP} - 2d_{ll} - 4d_{lP} - 2d_{PP})(p_K p_l)(p_l q));
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
|M|_{Low-3a-IR}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{2e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)^2} \\
&\times (a_0(p_l q)m_l^2 + 2(a_P - 2a_0)(p_K p_l)m_l^2 + (c_K - 2c_0 + 2c_P)(p_l q)^2 m_K^2 \\
&- 2c_K(p_K p_l)(p_l q)m_K^2 + 2(c_0 - c_P)(p_K p_l)(p_l q)(2p_K p_l + 2p_K q - p_l q) \\
&+ c_P m_l^2(p_K p_l)(p_l q));
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
|M|_{Low-3a-mag}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)^2} \\
&\times (-2a_P)(2(p_K p_l)m_l^2 + (p_l q)m_l^2 + 2(p_l q)^2 - 4(p_K p_l)(p_l q));
\end{aligned} \tag{C.16}$$

$$\begin{aligned}
|M|_{Low-3a-Low}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{2e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)} (2(a_P - a_0)(p_l q) \\
&- 2c_P(p_K p_l)(p_l q) + c_K((p_l q)m_K^2 - 2(p_K p_l)^2 - 2(p_K p_l)(p_K q)));
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
|M|_{Low-3b-IR}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)} \\
&\times (-2d_{KK}(p_l q)m_K^4 + 2(-d_K - 2d_P + 2d_{KP} + 2d_{lP} + 2d_{PP})(p_l q)^2 m_K^2 \\
&+ 2(-2d_l - 2d_P + d_{Kl} + d_{KP} + d_{ll} + 2d_{lP} + d_{PP})(p_l q)m_l^2 m_K^2 \\
&+ 4d_2(p_l q)m_K^2 + 8(d_K - d_{Kl} - d_{KP})(p_K p_l)(p_l q)m_K^2 + 2d_{KK}(p_K q)(p_l q)m_K^2 \\
&+ (2b_l + 2b_P - 3b_{ll} - 6b_{lP} - 3b_{PP})m_l^4 + 4(b_l - b_{ll} - b_{lP})(p_K q)m_l^2 \\
&+ 4(-b_0 + b_l + 3b_P - b_{lP} - 2b_{PP})(p_l q)^2 \\
&+ 4(d_0 - d_l - 2d_P + d_{lP} + d_{PP})(p_K p_l)(p_l q)^2 \\
&+ 4(2b_0 - 5b_l - 5b_P + 3b_{ll} + 6b_{lP} + 3b_{PP})(p_K p_l)m_l^2 \\
&+ 2(-2b_0 + 5b_l + 6b_P - b_{ll} - 6b_{lP} - 5b_{PP})(p_l q)m_l^2 \\
&+ 8(-d_0 + 2d_l + 2d_P - d_{ll} - 2d_{lP} - d_{PP})(p_K p_l)^2(p_l q) \\
&+ 2(-d_l - d_P + d_{ll} + 2d_{lP} + d_{PP})(p_K p_l)(p_l q)m_l^2 \\
&+ 8(b_0 - 2b_l - 3b_P + b_{ll} + 3b_{lP} + 2b_{PP})(p_K p_l)(p_l q) \\
&+ 2(d_P + d_{ll} + d_{lP})(p_K q)(p_l q)m_l^2 - 4b_2(m_l^2 - 2(p_K p_l) + (p_l q)) \\
&+ 4(-d_K + 2d_P + d_{Kl} - 2d_{lP} - 2d_{PP})(p_K p_l)(p_K q)(p_l q));
\end{aligned} \tag{C.18}$$

$$\begin{aligned}
|M|_{Low-3b-mag}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{e^2 \xi m_K^4}{\pi^2 (p_K q)(p_l q)} \\
&\times ((b_{ll} + 2b_{lP} + b_{PP})m_l^4 - 4b_2 m_l^2 + 4(b_{PP} - b_P)(p_l q)^2 + 8b_2(p_K p_l) \\
&+ 4(b_l + b_P - 2b_{ll} - 4b_{lP} - 2b_{PP})(p_K p_l)m_l^2 \\
&+ 4(b_P - b_{lP} - b_{PP})(p_K q)m_l^2 + 2(-b_l - b_P + 2b_{lP} + 2b_{PP})(p_l q)m_l^2);
\end{aligned} \tag{C.19}$$

$$\begin{aligned}
|M|_{Low-3b-Low}^2 &= \left(\frac{G}{\sqrt{2}} \sin \theta_c e f_+(0) \right)^2 \frac{-2e^2 \xi m_K^4}{\pi^2 (p_K q)} \\
&\times (d_{KK}(p_K p_l)m_K^2 - (d_{Kl} + d_{KP})m_l^2 m_K^2 + d_{Kl}(p_l q)m_K^2 \\
&+ (-2b_l - 2b_P + b_{ll} + 2b_{lP} + b_{PP})m_l^2 \\
&+ 2(-d_K + d_{Kl} + d_{KP})(p_K p_l)^2 + 2b_2 + 2d_2(p_K p_l) \\
&+ (-d_{ll} - 2d_{lP} - d_{PP})(p_K q)m_l^2 \\
&+ 2d_{KP}(p_K p_l)(p_K q) + 2(b_0 - b_l - 2b_P + b_{lP} + b_{PP})(p_l q) \\
&+ 2(-d_l - d_P + d_{ll} + 2d_{lP} + d_{PP})(p_K p_l)(p_l q).
\end{aligned} \tag{C.20}$$

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